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On the significance of the radial Newtonian gravitational force of the finite cylinder

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Abstract. An exact analytical solution of the radial Newtonian gravitational attraction of a finite cylinder is given and leads to a discussion of the nonlinear terms in the equation of motion of the pendulum in Heyl's determination of the constant of gravitation. The optimum shape of cylinder has a diameter to length ratio of $D_0/L_0 = 1.029\,282$, giving the maximum gravitational attraction of cylinders having the same mass and density.

1. Introduction

Nearly 200 years ago, Henry Cavendish (1798) used the torsion balance to measure the first physical constant—Newtonian gravitational constant—in history. His result compares excellently with modern measurements (Heyl 1930, Heyl and Chranowski 1942, Rose *et al* 1969). But it is well known that the gravitational constant G, which is the first to be discovered, is the poorest known with only three figures confirmed. It is not compatible with its significance in physics.

Different methods, torsion balance, beam balance, acceleration, etc have been used in laboratory determinations of the constant of gravitation and different kinds of attracting body have been used as well. A solid cylinder has been used as a suitable shape of attracting mass for the cylinder has many advantages in practice, easy to make and easy to measure the distances. But because of mathematical difficulties there is still no exact solution of the gravitational field of the cylinder. Heyl published the result of his extensive mathematical calculations of the radial gravitational force of the cylinder in the form of a polynomial to nearly ten pages. An analytical exact solution of the gravitational field of a cylinder would be of value not only because of the mathematical interest but also for the practical reasons. The polynomial formula, beside making the calculations very laborious, is inconvenient for the study of some important problems. For example, in the measurement of G, the shape of the attracting cylinder should be chosen so as to maximise the attraction of the test mass. It is a practical problem. Furthermore, one of the authors (Cook 1970) pointed out ten years ago that, in Heyl's experiment, the motion of the torsion pendulum under the attraction of the cylinders may be highly nonlinear, leading to systematic error in the reduction of the observation; the effects have not hitherto been explored in detail. All these works need an analytical formula for the field of the cylinder.

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Recently one of us (Y T Chen) used a dummy variables transformation to overcome the mathematical difficulty of the problem. An exact formula for the radial Newtonian gravitational force produced by a finite cylinder at any point is published in this paper, so that the nonlinear character of the behaviour of the pendulum in the field of two cylinders (as in Heyl's experiment) can be explored. A discussion of the optimal shape of the cylinder is given.

2. The radial attraction of a finite cylinder

Consider a cylinder of length L, radius R and density ρ. We shall consider the attraction at any arbitrary point, but for the convenience of the derivation, a point P in the base plane of the cylinder is first considered. Take cartesian coordinates with the x axis through point P. Let a be the radial distance of P from the centre line of the cylinder, so that the radial gravitational force due to the cylinder on the unit mass at the point P will be, (figure 1).

$$F_a^{(1)} = \frac{\partial}{\partial a}\left(-\int_{-R}^{R} \mathrm{d}y \int_{-\sqrt{(R^2-y^2)}}^{\sqrt{(R^2-y^2)}} \mathrm{d}x \int_0^L \mathrm{d}z \frac{G\rho}{\sqrt{y^2+z^2+(x-a)^2}}\right). \tag{1}$$

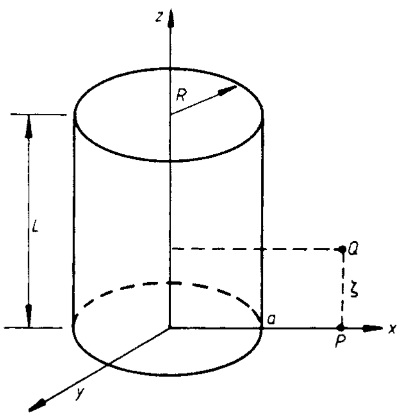

Figure 1.

On differentiating with respect to variable a and integrating with respect to z and x, we find

$$F_a^{(1)} = 2G\rho(-I_1+I_2) \tag{2}$$

where I_1 and I_2 are

$$I_1 = \int_0^R \ln\left(\frac{L+(L^2+R^2+a^2+2a\sqrt{R^2-y^2})^{1/2}}{L+(L^2+R^2+a^2-2a\sqrt{R^2-y^2})^{1/2}}\right) \mathrm{d}y \tag{3}$$

$$I_2 = \frac{1}{2}\int_0^R \ln\left(\frac{R^2+a^2+2a\sqrt{R^2-y^2}}{R^2+a^2-2a\sqrt{R^2-y^2}}\right) \mathrm{d}y. \tag{4}$$

Although there is only one variable in equation (3), two variables were used in the following transformation

$$\xi = 1 + \left(1 + \frac{R^2 + a^2}{L^2} + \frac{2aR}{L^2}\sqrt{1 - y^2/R^2}\right)^{1/2} \tag{5}$$

$$\eta = 1 + \left(1 + \frac{R^2 + a^2}{L^2} - \frac{2aR}{L^2}\sqrt{1 - y^2/R^2}\right)^{1/2}. \tag{6}$$

Thus

$$I_1 = \int_{\xi_1}^{\xi_2} -\frac{[B^2 - A^2 + 2A(\xi - 1)^2 - (\xi - 1)^4]^{1/2}}{\xi}\,\mathrm{d}\xi + \int_{\eta_1}^{\eta_2} \frac{[B^2 - A^2 + 2A(\eta - 1)^2 - (\eta - 1)^4]^{1/2}}{\eta}\,\mathrm{d}\eta \tag{7}$$

where

$$A = 1 + (R^2 + a^2)/L^2 \qquad B = 2Ra/L^2$$

and

$$\xi_1 = 1 + (A + B)^{1/2} \qquad \xi_2 = 1 + A^{1/2}$$

$$\eta_1 = 1 - (A - B)^{1/2} \qquad \eta_2 = 1 + A^{1/2}.$$

The two integrals in (7) have the same form of integrand but different limits. Because the two limits ξ_2 and η_2 are the same, we can regard ξ and η as dummy variables, and write equation (7) as

$$I_1 = \int_{1+\sqrt{(A-B)}}^{1+\sqrt{(A+B)}} \frac{[B^2 - A^2 + 2A(t - 1)^2 - (t - 1)^4]^{1/2}}{t}\,\mathrm{d}t. \tag{8}$$

Although we cannot give the analytical relationship between the variables y in equation (3) and t in equation (8), the value of the integral (8) must be independent of the variable t, and so it is equivalent to equation (3). In this way, equation (2) becomes

$$\frac{F_a^{(1)}}{2G\rho} \equiv C_y(L, R, a)$$

$$= \frac{L^2}{a\sqrt{1 + [(R + a)/L]^2} + \sqrt{1 + [(R - a)/L]^2}}$$

$$\times \Bigg\{ \Bigg[1 + 2\frac{R^2 + a^2}{L^2} + \sqrt{1 + [(R - a)/L]^2} \times \left(\sqrt{1 + [(R + a)/L]^2} + \frac{R^2 + a^2}{L^2} \right)$$

$$+ \left(\frac{R^2 - a^2}{L^2} \right)^2 \frac{1}{1 - \sqrt{1 + [(R - a)/L]^2}} \Bigg] K(k)$$

$$- \frac{1}{2}[1 + [(R + a)/L]^2 + \sqrt{1 + [(R - a)/L]^2}]^2 E(k)$$

$$- 2\frac{R^2 + a^2}{L^2}\sqrt{1 + [(R - a)/L]^2}\prod(\tfrac{1}{2}\pi, k, k)$$

$$+ 2\frac{(R + a)^2}{L^2}\sqrt{1 + [(R - a)/L]^2}\prod(\tfrac{1}{2}\pi, \alpha^2, k) \Bigg\} + I_0 \tag{9}$$

where

$$I_0 = \begin{cases} \pi R^2/2a & \text{when } a \geqslant R \\ \frac{1}{2}\pi a & \text{when } a \leqslant R \end{cases}$$

$$k = \frac{\sqrt{a + [(a+R)/L]^2} - \sqrt{1 + [(a-R)/L]^2}}{\sqrt{1 + [(a+R)/L]^2} + \sqrt{1 + [(a-R)/L]^2}} \tag{10}$$

$$\alpha^2 = \frac{1 - \sqrt{1 + [(a-R)/L]^2}}{1 + \sqrt{1 + [(a+R)/L]^2}}.$$

$K(k)$, $E(k)$, $\Pi(\frac{1}{2}\pi, k, k)$ and $\Pi(\frac{1}{2}\pi, \alpha^2, k)$ are the elliptical integrals of the first, second and third kind respectively.

Equation (9) is the formula for the point in the end plane of the cylinder but with the superposition principle of gravitation the formula for an arbitrary point is

$$F_a/2G\rho = C_y(L - \zeta, R, a) + C_y(\zeta, R, a) \tag{11}$$

where L is the length of the cylinder and ζ is the distance of the point Q from the end plane of the cylinder as shown in figure 1.

In Heyl's experiment, the test mass was in the middle plane of the cylinder (see figure 2) so the attraction, according to equation (11) is

$$F_a^{(2)} = 2F_a^{(1)} = 4G\rho C_y(\tfrac{1}{2}L, R, a). \tag{12}$$

If $\zeta < 0$ in figure 1, according to the superposition principle, equation (11) will become

$$F_a/2G\rho = C_y(L + |\zeta|, R, a) - C_y(|\zeta|, R, a). \tag{13}$$

The radial Newtonian gravitational force of any complex co-axial cylinders will not be difficult to obtain from the following sum

$$F_a = 2G\rho \sum_i C_y(L_i, R_i, a). \tag{14}$$

3. The implication for Heyl's experiment

Heyl measured the period of the torsional pendulum in the near position and far position from a pair of cylinders (see figure 2). He had paid great attention to the details of his experiment, so that although it was performed a long time ago (1930 and 1942) his result is still regarded as one of the most accurate results and quoted very often in the literature.

As a check on Heyl's approximate formula used in his calculations, the following comparison is an example.

To illustrate the convergence of his series, Heyl calculated the attraction of a cylinder upon a point at the centre of its lateral surface,

$$a = R = \tfrac{1}{2}L = 10 \text{ cm}.$$

He kept eleven terms of his formula, the result is

$$F_a^{(2)}/\pi a G\rho = 1.535\,5797.$$

According to equations (12) and (9), this number is

$$\frac{F_a^{(2)}}{\pi a G\rho}=\frac{4}{\pi}\left[\sqrt{5}\,K(k)-\frac{2}{\sqrt{5}-1}E(k)\right]=1.535\,5784.$$

We notice that this result is independent of the numerical value of a and R.

It is clear from equation (9) that the motion of a test particle in the field is highly nonlinear, even though the displacement is very small. Consider the equation of motion of a pendulum with the cylinders in the near position (figure 2).

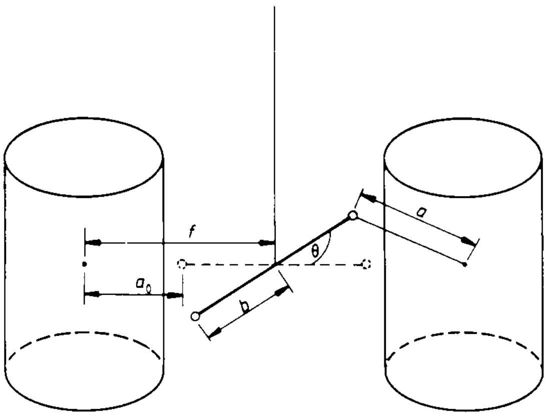

Figure 2.

For simplicity the mass of the beam is neglected and the mass m of the ball is taken to be unity. In the ideal case without damping, the Lagrangian function of the pendulum is

$$\mathscr{L}=\tfrac{1}{2}I\dot{\theta}^2-V(\theta)-\tfrac{1}{2}\tau\theta^2 \tag{15}$$

where I is the moment of inertia of the whole pendulum, τ is the modulus of torsion of the filament, assumed constant with respect to θ. $V(\theta)$ is the potential energy in the field produced by the cylinders and can be obtained from the result in § 2 and the geometrical relation to figure 2. Then

$$\partial V/\partial\theta=8G\rho C_y(\tfrac{1}{2}L,R,a)\,(bf/a)\sin\theta. \tag{16}$$

Thus the Lagrangian equation of motion becomes

$$I\ddot{\theta}+\tau\theta+8G\rho C_y(\tfrac{1}{2}L,R,a)\,(bf/a)\sin\theta=0. \tag{17}$$

The function $C_y(\frac{1}{2}L,R,a)$ is a function of θ. To show the nonlinear character of equation (17) when θ is much less than 1, we expand $C_y(\frac{1}{2}L,R,a)$ around the equilibrium point a_0

$$\begin{aligned}C_y(\tfrac{1}{2}L,R,a)&=C_y(\tfrac{1}{2}L,R,a_0)+\left[\frac{\partial C_y(\frac{1}{2}L,R,a)}{\partial a}\right]_{a=a_0}(a-a_0)\\&=C_y(\tfrac{1}{2}L,R,a_0)+\left[\frac{\partial C_y(\frac{1}{2}L,R,a)}{\partial a}\right]_{a=a_0}\frac{bf}{2a_0}\theta^2\end{aligned} \tag{18}$$

so that the equation of motion with damping will be

$$I\ddot{\theta}+2m\dot{\theta}+\tau\theta+\alpha\theta+\beta\theta^3+\gamma\theta^5+\ldots=0 \tag{19}$$

where

$$\alpha=8G\rho\,(bf/a_0)\,C_y(\tfrac{1}{2}L,R,a_0)$$

$$\beta=-4G\rho\frac{bf}{a_0}\left\{\left(\frac{1}{3}+\frac{bf}{a_0^2}\right)C_y(\tfrac{1}{2}L,R,a_0)-\frac{bf}{a_0}\left[\frac{\partial C_y(\tfrac{1}{2}L,R,a)}{\partial a}\right]_{a=a_0}\right\}. \tag{20}$$

Comparing the two coefficients α and β, the ratio is

$$\left|\frac{\beta}{\alpha}\right|=\frac{1}{2}\left\{\frac{1}{3}+\frac{bf}{a_0^2}-\frac{bf}{a_0}\frac{1}{C_y(\tfrac{1}{2}L,R,a_0)}\left[\frac{\partial C_y(\tfrac{1}{2}L,R,a)}{\partial a}\right]_{a=a_0}\right\}. \tag{21}$$

It is very easy to prove that

$$C_y(\tfrac{1}{2}L,R,a_0)>0 \qquad \left[\frac{\partial C_y(\tfrac{1}{2}L,R,a)}{\partial a}\right]_{a=a_0}<0$$

so that the third term in the bracket of equation (21) is positive as well. Equation (19) shows that the period of the oscillation is a function of the amplitude and damping factor, but in the case of small amplitude and weak damping, we can assume equation (19) has the following solution

$$\theta=A\exp(-mt/I)\cos\omega t+B\exp(-mt/I)\cos 3\,\omega t \tag{22}$$

where A, B are constant, and

$$B<A\ll 1 \qquad mt/I\ll 1.$$

ω is the frequency of the pendulum at point θ. Insert the solution of equation (22) into the differential equation, then we have

$$\omega^2=\frac{\tau+\alpha}{I}+\frac{3\beta}{4I}\theta_0^2-\frac{m^2}{I^2}+\ldots \tag{23}$$

where θ_0, is the amplitude, that is, the period–amplitude–damping relation can be expressed by

$$T=T_0\left[1-\frac{3}{8}\cdot\frac{\beta}{\tau+\alpha}\theta_0^2+\frac{1}{2}\cdot\frac{m^2}{I(\tau+\alpha)}+\ldots\right] \tag{24}$$

where

$$T_0=2\pi\sqrt{I/(\tau+\alpha)}$$

is the period for the zero amplitude without damping.

4. Optimum shape of the cylinder

The aim of this section is to decide the shape of the cylinder which can produce the maximum radial attraction among all cylinders with the same mass and density. The optimum shape of the cylinder is dependent upon the value of a, but because in the experiment of measuring G, the distance of test mass to the attracting mass should be set as close as possible, we are only interested in the case $a\sim R$. It may be proved that, in case of $a\geqslant R$, $C_y(L,R,a)$ has no stationary points with respect to a. Thus we choose the point $a = R$ as the point at which to compare the magnitude of force for all the cylinders, because it is the point of maximum field for every cylinder.

If $a = R$, equation (9) can be simplified into

$$F_a^{(1)} = R\left[\frac{1-k^2}{k}K(k) - \frac{1-k}{k}E(\tfrac{1}{2}\pi, k)\right]2G\rho. \tag{25}$$

To find the ratio R/L, for which the expression (25) attains its maximum value under the following condition

$$R^2L = \text{constant} = Q \tag{26}$$

we use the Lagrangian method of undetermined multipliers and find the equation determining the stationary point of (25) subject to (26) to be

$$\frac{E(\frac{1}{2}\pi, k) - K(k)}{k} + kK(k) - E(\tfrac{1}{2}\pi, k) + 48\frac{R^6}{Q^2}\cdot\frac{K(k) - E(\frac{1}{2}\pi, k)}{(1+\sqrt{1+4R^6/Q^2})(1-\sqrt{1+4R^6/Q^2})^2} = 0 \tag{27}$$

and according to equation (10)

$$k = \frac{\sqrt{1+4R^6/Q^2}-1}{\sqrt{1+4R^6/Q^2}+1}. \tag{28}$$

The solution of equation (27) having physical significance is

$$R = 1.009\,667(Q)^{1/3}$$

or according to equation (26)

$$R = 1.029\,282\,56L$$

It is not difficult to show that this solution gives the maximum value of equation (25). Similarly by the superposition principle of gravitation, if we put the test mass at the middle plane of the cylinder, the result is

$$D_0/L_0 = 1.029\,282$$

where D_0, L_0 are the diameter and length of the cylinder.

Further calculation has shown that this shape of cylinder can produce a gravitational attraction even greater than the maximum attraction produced by a sphere with the same mass and density, so that, in practice, we should choose the cylinder with a diameter to length ratio of roughly 1 which can give more gravitational efficiency.

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